

Last time: vector functions

$$\vec{r}(t) = \langle f(t), g(t) \rangle \text{ (2D)} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \text{ (3D)}$$

Input: number $t \rightarrow$ Output: 2 numbers / 3 numbers

Today: Functions of several variables

Input: 2 numbers / 3 numbers $\rightarrow$ Output: a number.

Usually written like...

$f(x,y)$: Takes 2 numbers x, y as inputs

Gives a value $f(x,y)$ as an output

$f(x,y,z)$: Takes 3 numbers x, y, z as inputs

Gives a value $f(x,y,z)$ as an output

Example: $f(x,y) = x+y$, $f(x,y) = x^2 + \sin y$, $f(x,y) = e^{xy}$, ...

$$f(1,2) = 1+2=3 \quad f(0,\pi) = 0^2 + \sin \pi = 0 \quad f(1,3) = e^3$$

The expression for $f(x,y)$ may make sense at certain parts of the xy -plane.

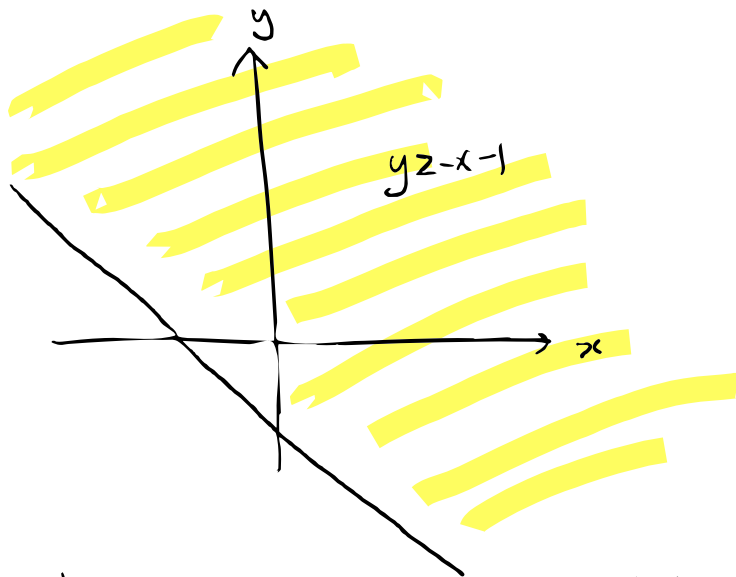
The region over which the expression makes sense is called the domain. The domains in this case are much more complicated than functions of one variable.

This is because the 2D/3D regions are more complicated. When drawing the domain, we draw solid boundary lines if they are included, dotted if not included.

Example $f(x, y) = \sqrt{x+y+1}$

The expression makes sense if $x+y+1 \geq 0$, or $x+y \geq -1$.

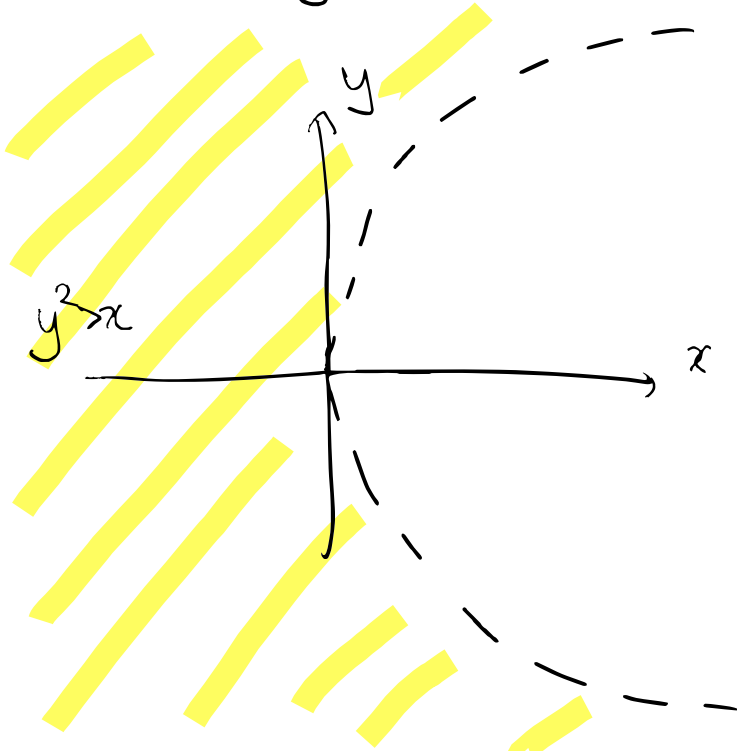
The domain is $\{x+y \geq -1\}$.



Boundary line is $y = -x - 1$ and is included in the domain.

Example $f(x, y) = x \ln(y^2 - x)$ is defined if $y^2 - x > 0$.

The domain is $\{y^2 > x\}$.



The dotted boundary line is $y^2 = x$ and is not included.

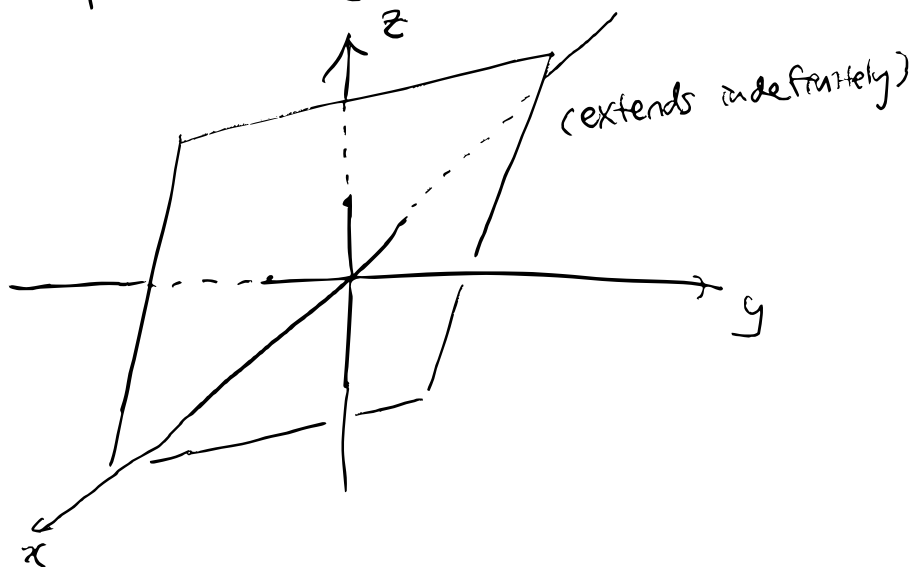
For a function of two variables $f(x, y)$, you can sketch the graph of $z = f(x, y)$.

This is the collection of points

$$\{(x, y, f(x, y))\}.$$

This is a surface in 3D.

Example The graph of $z=f(x,y)$, where $f(x,y)=x+y+1$, is the plane $z=x+y+1$.



Example

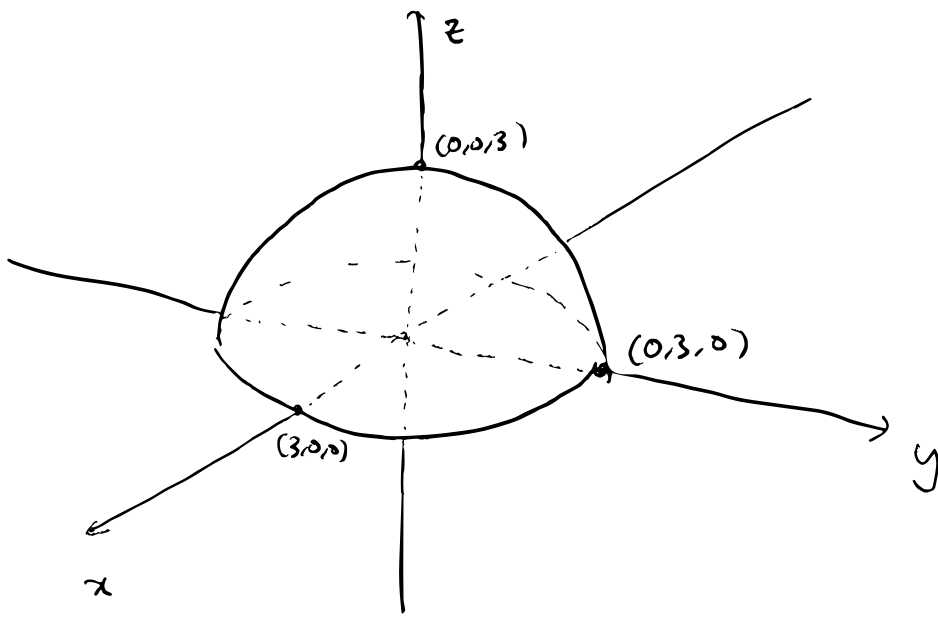
The graph of $z=f(x,y)$, where $f(x,y)=\sqrt{9-x^2-y^2}$, satisfies $z=\sqrt{9-x^2-y^2}$, or $z^2=9-x^2-y^2$, or

$$\boxed{x^2+y^2+z^2=9.}$$

This is the equation of a sphere but note that the graph does not depict the whole sphere, because

$z=\sqrt{9-x^2-y^2}$ is always $z \geq 0$.

The graph is thus the upper half of the sphere, including the boundary (=equator).



Just like the domains of vector functions, there are some properties that you can check for the domains of functions of several variables.

- **Closed**: When the domain is defined using only $\geq$, $\leq$, $=$ in the expression (NO $>$, $<$, $\neq$)
- **Bounded**: When the domain does not go to infinity and "stay bounded"
- **Compact**: When the domain is both closed and bounded.

Implicit functions

A function can be defined implicitly using an implicit equation involving x, y, z .

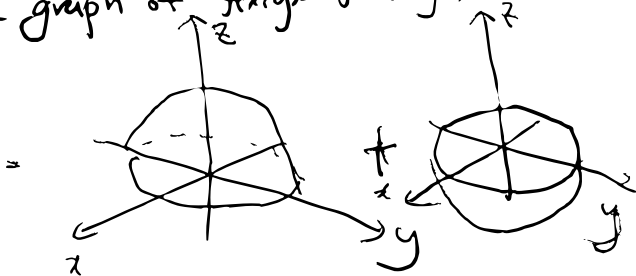
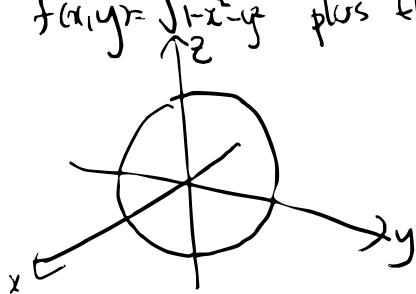
Ex The implicit equation $x^2 + y^2 + z^2 = 1$ can be regarded as defining an implicit function z in terms of x, y . Namely,

$$z(x, y) = \{ \text{The values of } z \text{ such that } x^2 + y^2 + z^2 = 1 \}.$$

Implicit functions have a peculiar feature that there might be multiple different possible values for z given x, y . In the example of $x^2 + y^2 + z^2 = 1$, there are 2 possible values, $\pm \sqrt{1 - x^2 - y^2}$, and implicit function means you are considering both functions together.

The graph of $x^2 + y^2 + z^2 = 1$ therefore is the graph of

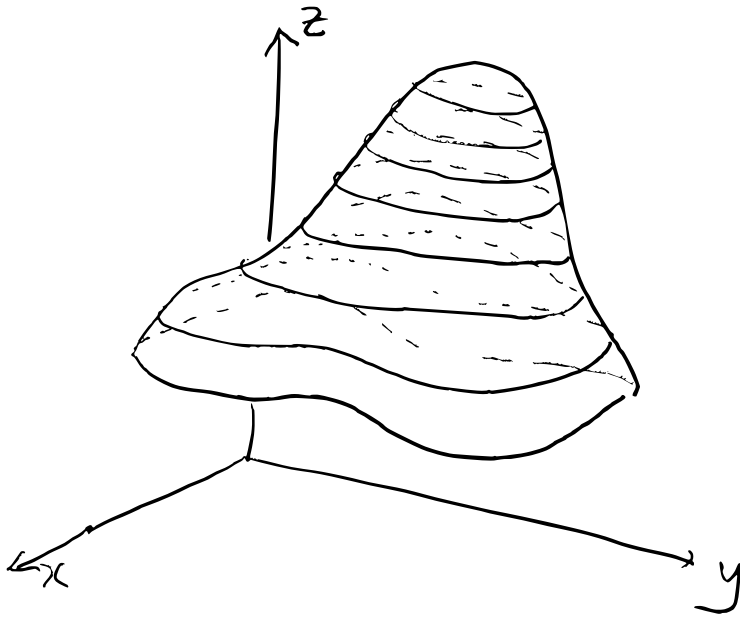
$f(x, y) = \sqrt{1 - x^2 - y^2}$ plus the graph of $f(x, y) = -\sqrt{1 - x^2 - y^2}$.



Horizontal traces and level curves

Since it's difficult to draw a 3D graph on paper, we often put some indicators on the graph to better depict the 3D object.

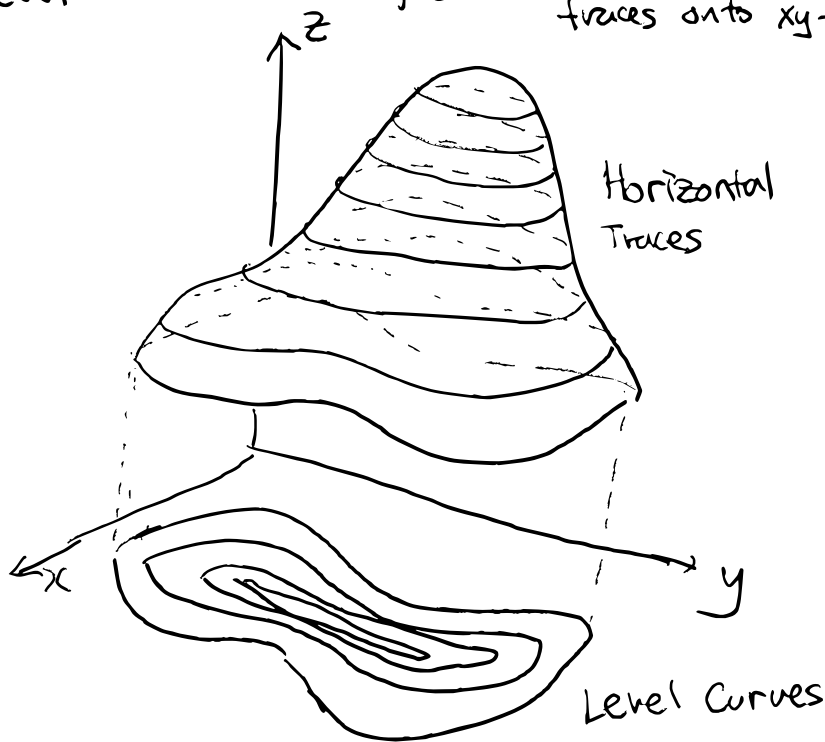
Definition The horizontal trace of level k is the curve in 3D that is the " $z=k$ slice" of the graph. Namely, given an equation for the graph, you set $z=k$. The curve has a fixed z -coordinate, k .



You can even try to encode the data of $f(x,y)$ purely on the xy -plane using this idea.

Definition The level curve of level k is the curve on 2D defined by setting $z=k$ from the equation of the graph. For example, if the graph is $z=f(x,y)$, The level curve of level k is $\{(x,y) \mid f(x,y)=k\}$.

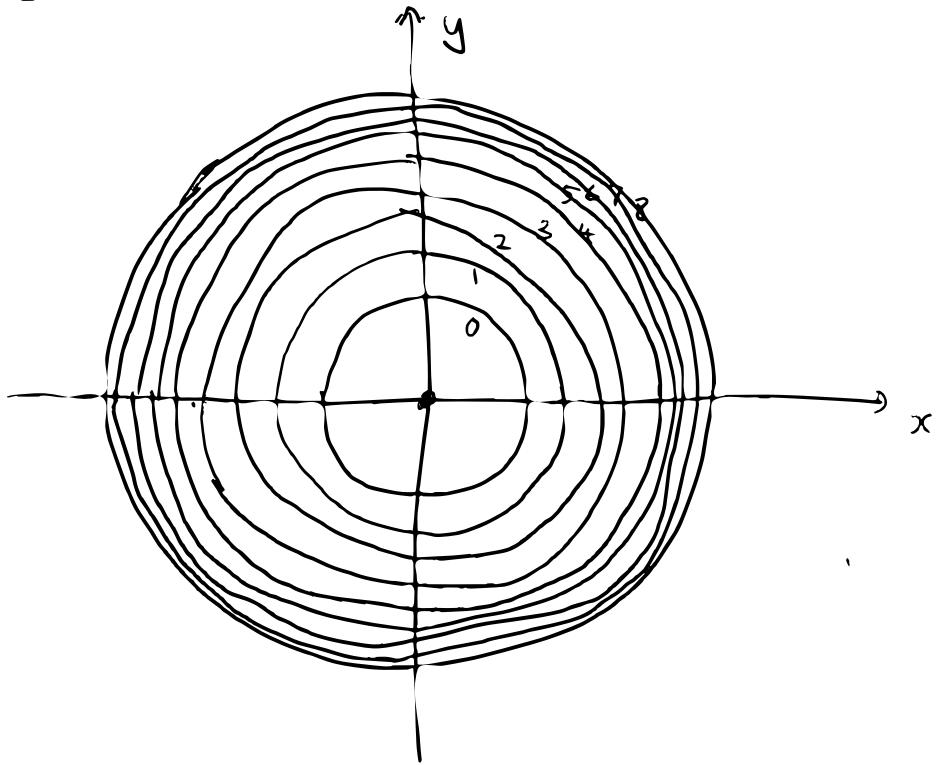
The level curves are the projections of the horizontal traces onto xy -plane!



A collection of level curves on 2D is called a contour map. Example) Topographic maps, weather forecast maps.



Example From the graph of $z = \sqrt{9 - x^2 - y^2}$,
the contour map looks like:



The numbers on the level curves indicate the levels.

Or, you may use different colors for different level curves.